ANDHRA UNIVERSITY
SCHOOL OF DISTANCE EDUCATION
M.A. / M.SC., Mathematics (Final Year)
ASSIGNMENTS
Measure Theory and Functional Analysis
Answer any five of the following. All question carry equal marks.

5x3=15 Marks

1. (a) Define outer measure and prove that the outer measure of an interval is its length.
   (b) Define measurable set and prove that the interval \((a, \infty)\) is measurable for any real number \(a\).

2. (a) State and prove bounded convergence theorem.
   (b) Let \(f\) be a non negative function which is integrable over a set \(E\). Then prove that for given \(\epsilon > 0\) there is a \(\delta > 0\) such that for every set \(A \subseteq E\) with \(m(A) < \delta\) we have
   \[
   \int_A f < \epsilon.
   \]

3. (a) i) Define \(\sigma\) – algebra.
   ii) Let \(B\) be a \(\sigma\) – algebra. If \(E_i \in B\) and \(\mu\left(\bigcup E_i\right) < \infty\) with \(E_i \supseteq E_{i+1}\), then prove that
   \[
   \mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n \to \infty} \mu(E_n)
   \]

   (b) State and prove Fatou’s lemma

4. (a) i) Define signed measure
   ii) State and prove Hahn decomposition theorem.
   (b) State and prove Radon - Nikodym theorem.

5. (a) i) Define normed linear space and Banach space.
   ii) Let \(M\) be a closed linear subspace of a normed linear space \(N\). If the norm of a coset \(x + M\) in the quotient space \(N/M\) is defined by
   \[
   \|x + M\| = \inf \{\|x + m\| : m \in M\}
   \]
   then prove that \(N/M\) is a normed linear space. Further, if \(N\) is a Banach space, then prove that \(N/M\) is also a Banach space.
(b) State and prove the Hahn-Banach theorem.

6. (a) State and prove the open mapping theorem.
(b) State and prove the uniform boundedness theorem.

7. (a) i) Define Hilbert space.
   ii) State and prove Schwarz inequality.
(b) Prove that a closed convex subset C of a Hilbert space $H$ contains a unique vector of smallest norm.

8. (a) Let $H$ be a Hilbert space, and let $f$ be an arbitrary functional in $H^*$. Then prove that there exists a unique vector $y$ in $H$ such that $f(x) = \langle x, y \rangle$ for every $x$ in $H$.
(b) Prove that the adjoint operation $T \rightarrow T^*$ on $B(H)$ has the following properties.
   i) $(T_1 + T_2)^* = T_1^* + T_2^*$
   ii) $(\alpha T)^* = \overline{\alpha} T^*$
   iii) $(T_1 T_2)^* = T_2^* T_1^*$
   iv) $T^{**} = T$
   v) $\|T^*\| = \|T\|$
   vi) $\|T^* T\| = \|T\|^2$
1. (a) If $G$ is open and connected subset of the complex plane and if $f: G \rightarrow \mathbb{C}$ is differentiable with $f'(z) = 0$ for all $z$ in $G$ then prove that $f$ is constant.
(b) State and prove Liouville's theorem and deduce fundamental theorem of Algebra.

2. (a) Show that the function $f: \mathbb{C} \rightarrow G$ defined by $f(z) = |z|^2$ is differentiable only at the origin.
(b) If $G$ is a region and $f: G \rightarrow \mathbb{C}$ is a analytic function such that there is a point $a$ in $G$ with $|f(z)| \leq |f(a)|$ for all $z$ in $G$. Then prove that $f$ is a constant.

3. (a) State and prove open mapping theorem.
(b) State and prove Laurent series development theorem.

4. (a) State and prove Hadamard Three circles theorem
(b) State and prove third version of Maximum modulus theorem

5. (a) State and prove Riemann mapping theorem
(b) State and prove Weierstrass factorisation theorem.

6. (a) Discuss factorisation of sine function
(b) Derive the Gauss Formula and prove Bohr Moller Theorem.

7. (a) State and prove Range's theorem
(b) Discuss Swartz reflection principle.

8. (a) State and prove Monodromy theorem
(b) State and prove Mittag-Leffler's theorem.
1. (a) Define Euler Totient function

Prove that \( \sum_{d|n} \phi(d) = n \)

(b) Prove that \( \phi(n) = n \sum_{d|n} \frac{M(d)}{d} \) for all positive integers \( n \).

(c) Prove that \( \phi(mn) = \phi(m) \phi(n) \) if \( (m,n) = 1 \)

2. (a) Let \( f \) and \( g \) be two arithmetical functions. Define the Dirichlet product of \( f \) and \( g \).

Prove that for an arithmetical function \( f \) with \( f(1) \neq 0 \) there is a unique arithmetical function \( f^{-1} \) such that \( f \ast f^{-1} = f^{-1} \ast f = I \). Moreover \( f^{-1} \) is given by

\[
 f^{-1}(1) = \frac{1}{f(1)} \quad \text{and} \quad f^{-1}(n) = \frac{-1}{f(1)} \cdot \sum_{d|n} f\left(\frac{n}{d}\right) f^{-1}(d)
\]

(b) State and prove Möbius inversion formula.

(c) Define multiplicative function. Prove that if two of the three arithmetical function \( f, g \) and \( f \ast g \) are multiplicative then the other is multiplicative.

3. (a) State and prove Euler Summation Formula

(b) Prove that if \( s > 0 \) and \( s \neq 1 \) then for any \( x \geq 1 \)

\[
 \sum_{n \leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \psi(s) + O\left(\frac{1}{x^s}\right)
\]

(c) For \( x \geq 1 \) prove that

\[
 \sum_{n \leq x} d(n) = x \log x + O(1)
\]

4. (a) Let \( x > 0 \). Prove that \( 0 \leq \frac{\psi(x)}{x} - \frac{\theta(x)}{x} \leq \frac{(\log x)^2}{2 \sqrt{x \log 2}} \)

(b) For \( x \geq 2 \) prove that
\[ \theta(x) = \pi(x) \log x - \int_{\frac{x}{2}}^{x} \frac{\pi(t)}{t} \, dt \text{ and } \]

\[ \pi(x) = \frac{\theta(x)}{\log x} + \int_{\frac{x}{2}}^{x} \frac{\theta(t)}{t \log t} \, dt \]

(c) Prove that the following statements are equivalent

i) \( \lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1 \)

ii) \( \lim_{x \to \infty} \frac{\theta(x)}{x} = 1 \)

iii) \( \lim_{x \to \infty} \frac{\psi(x)}{x} = 1 \)

5. (a) For \( n \geq 2 \), prove that the following inequalities hold for the functions \( \pi(x) \) and \( \log n \).

(b) State and prove Euler - Fermat theorem

(c) State and prove Lagrange's theorem

6. (a) State and prove Chinese - remainder theorem

(b) Prove that a finite abelian group \( G \) of order \( n \) has exactly \( n \) distinct characters.

(c) State and prove the orthogonality relations for characters.

7. (a) Prove that there are infinitely many primes of the form (i) \( 4n - 1 \) and (ii) \( 4n + 1 \)

(b) State and prove Dirichlet theorem.

(c) Let \( P \) be an odd prime then prove that every reduced residue system modulo \( P \) contains exactly \( \frac{(P-1)}{2} \) quadratic residues and \( \frac{(P-1)}{2} \) quadratic non residues modulo \( P \).

8. (a) Let \( P \) be an odd prime. Then prove that for all integers \( n, (n/P) = n^{\frac{P-1}{2}} \)

(b) State and prove Gauss Lemma

(c) State and prove quadratic reciprocity law.
ANDHRA UNIVERSITY  
SCHOOL OF DISTANCE EDUCATION  
M.A. / M.SC., Mathematics (Final Year)  
ASSIGNMENTS  
Lattice Theory  

Answer any five of the following. All question carry equal marks.  

5x3=15 Marks

1. (a) Prove that if a partially ordered set $P$ satisfies the minimum condition, then for any $x \in P$, there exists at least one minimal element $m$ of $P$ such that $m \leq x$.  

(b) Let $P$ be a partly ordered set of locally finite length and satisfies the Jordan - Dedi kind chain condition. If $P$ contains an element $u$ such that $\inf \{u, x\}$ exists for all $x \in P$ then prove that dimension function can be defined on $P$.

2. (a) Prove that every uniquely complemented lattice is weakly complemented.  

(b) Prove that in a complemented lattice every join prime element except the least one is an atom of the lattice.

3. (a) Prove that every order preserving mapping $f$ of a complete lattice $L$ into itself has a fixed element; that is, there exists an element in $L$ such that $f(a)=a$.  

(b) Prove that every element of a compactly generated lattice can be represented on a meet of completely meet irreducible element.

4. (a) Prove that a lattice $L$ is modular if and only if a sublattice of $L$ is isomorphic to the lattice $N_3$ given by the following diagram.

(b) Let $L$ be a lattice in which every element has a unique irredundant irreducible meet representation. Then prove that $L$ satisfies the lower covering condition.

5. (a) Prove that every complete Boolean algebra is infinitely distributive.  

(b) Let $(B, \land, \lor)$ be a boolean algebra with 0 and 1 as the least and greatest elements and with $a'$ as the complement of $a$. For any $a, b \in B$, define $a+b=(a \land b') \lor (a' \land b)$ and $ab=a \land b$. Then prove that $(B, +, \cdot)$ is a Boolean
ring in which 0 and 1 are the additive and multiplicative identities respectively.

6. (a) Let $L$ be a lattice then prove that every ideal of $L$ is a principal ideal if and only if $L$ satisfies the maximum condition.

(b) Prove that a lattice is distributive if and only if it is isomorphic to a ring of sets (that is, a sublattice of the lattice $P(X)$ of a suitable set $X$).

7. (a) Prove that the congruence lattice $f(A)$ of any algebra $A$ is compactly generated.

(b) Prove that the congruence lattice $K(b)$ of any lattice $L$ is distributive, moreover, it is infinitely meet distributive.

8. (a) Let $L$ be a lattice bounded below. Then prove that every ideal of $L$ is the Kernel of a congruence relation on $L$ if and only if $L$ is distributive.

(b) Prove that every algebra can be represented as a union of sub direct irreducible algebra.
1. (a) Let $A$ be a ring $\neq 0$. Then prove that the following are equivalent.
   i) $A$ is a field
   ii) The only ideals of $A$ are $A$ and $\{0\}$.
   iii) Every homomorphism of $A$ into a non zero ring $B$ is injective
   (b) Prove that the nil radical of $A$ is the intersection of all prime ideals of $A$.

2. (a) Let $I_1, I_2, ..., I_n$ be ideals of a ring $A$.

   Define $\phi: A \to \pi A/I_i$ by $\phi(x) = (x+I_1, x+I_2, ..., x+I_n)$. Then prove the following.
   i) $I_i, I_j$ are co prime for $i \neq j$ implies $\pi I_i = \cap I_i$
   ii) $\phi$ is a homomorphism
   iii) $\phi$ is a surjective if and only if $I_i, I_j$ are coprime for $i \neq j$
   iv) $\phi$ is an injective if and only if $\cap_{i=1}^n I_i = 0$

   (b) State and prove Nakayama's lemma.

3. (a) Let $\xymatrix{M' \ar[r]^{f} & M \ar[r]^{g} & M'' \ar[r] & 0}$ be an exact sequence of $A-$ modules and homomorphism and let $N$ be any $A-$ module. Then prove that the sequence.

   $\xymatrix{M' \otimes N \ar[r]^{f \otimes 1} & M \otimes N \ar[r]^{g \otimes 1} & M'' \otimes N \ar[r] & 0}$ (where $1$ denotes the identity mapping of $N$) is exact.

   (b) Prove that, for any $A-$ module $M$, the $S^{-1} A-$ modules $S^{-1} M$ and $S^{-1} A \otimes_A M$ are isomorphic.
4. (a) Let $A$ be a ring and $M$ be an $A$-module. Then prove that the following are equivalent.
   i) $M = 0$
   ii) $M_p = 0$ for all prime ideals $P$ of $A$
   iii) $M_m = 0$ for all maximal ideals $m$ of $A$.

   (b) Let $S$ be a multiplicative set in a ring $A$ and $m$ be finitely generated $A_m$-module.
   Then prove that $S^{-1}\left(A_m(M)\right) = A_m(S^{-1}M)$.

5. (a) State and prove first uniqueness theorem.

   (b) Let $A$ be a subring of a ring $B$ and $x \in B$. Then prove that the following are equivalent to each other.
   i) $x$ is integral over $A$.
   ii) $A[x]$ is finitely generated $A$-module
   iii) $A[x]$ is contained in a subring $C$ of $B$ such that $C$ is a finitely generated as an $A$-module
   iv) There exists a faithful $A[x]$-module $M$ which is finitely generated as an $A$-module.

6. (a) Let $B$ be an integral domain and $A$ be a subring of $B$. Suppose $B$ is integral over $A$.
   Then prove that $B$ is field if and only if $A$ is a field.

   (b) State and prove going-down theorem.

7. (a) Let $A$ be a subring of an integral domain $B$ and $B$ be finitely generated over $A$. Let $0 \neq u \in B$. Then prove that there exists $0 \neq u \in A$ satisfying the following. For any algebraically closed field $K$ and for any homomorphism $f$ of $A$ into $K$ such that $f(u) \neq 0$, there is an extension of $f$ to a homomorphism $g : B \to K$ such that $g(u) \neq 0$.

   (b) Let $0 \to M \xrightarrow{f} M \xrightarrow{g} M^* \to 0$ be an exact sequence of $A$-module. Then prove that the following hold.
   i) $M$ is Noetherian if and only if $M$ and $M^*$ are Noetherian
   ii) $M$ is Artinian if and only if $M$ and $M^*$ are Artinian.

8. (a) Let $A$ be a Noetherian ring. Then prove that the polynomial ring $A[x]$ is also Noetherian.

   (b) Let $A$ be a local domain. The prove that $A$ is a discrete valuation ring if and only if every non-zero fractional ideal of $A$ is invertible.
1. (a) Suppose an elastic string is fixed at two distinct end points. If the weight is attached between these points, find the integral representation for the displacement of the string due to the complete weight distribution, assuming that the tension in the string is uniform.

(b) Find the eigen values $\lambda_n$ and eigen functions $y_n$ for $y'' + \lambda y = 0$ with the boundary condition $y(0) = 0, y(a) = 0$.

2. (a) Transform the differential equation.

$$y'' + 2xy' + y = 0$$

satisfying $y(0) = 1, y'(0) = 0$ into an integral equation.

(b) Find the integral equation for the problem defined by

$$\frac{d^2y}{dx^2} + 4y = f(x), 0 \leq x \leq \frac{\pi}{2}$$

satisfying $y(0) = 0$ and $y'\left(\frac{\pi}{2}\right) = 0$.

3. (a) Solve.

$$3x^2 + 4x = \int_0^x (6x^2 + 4xy) \varphi(y) dy$$

(b) Find the eigen values and eigen functions of the system

$$\varphi(x) = \lambda \int_0^1 (1 + xt) \varphi(t) dt, 0 \leq x \leq 1$$

4. (a) If $\varphi(x) = \lambda \int k(x, y) \varphi(y) dy$, where $k$ is a Hermitian kernel, has eigen values then show that they are all equal.

(b) If $k$ is Hermitain then prove that the eigen functions corresponding to different eigen values are orthogonal.

5. (a) Solve the integral equation

$$\varphi(x) = 3 \int_0^x \cos(x - y) \varphi(y) dy + e^x \text{ with } \varphi(0) = 1.$$
(b) Solve the integral equation
\[ \varphi(x) = \lambda \int_0^x e^{i(x-y)} \varphi(y) \, dy + f(x) \] by using the resolvent kernel method.

6. (a) Solve the integral equation
\[ \varphi(x) = x^3 + \int_0^x e^{x(y-x)} \varphi(y) \, dy \] by using the method of Laplace transforms.

(b) Solve the integral equation
\[ \varphi(x) = \int_0^1 \frac{\varphi(y)}{1+y} \, dy \] by using Picard's method.

7. (a) Write Picard's method for the existence, and hence to find solution of non-linear Volterra equation of second kind
\[ \varphi(x) = f(x) + \lambda \int_0^x F(x, y, \varphi(y)) \, dy. \]

(b) Find first and second approximation in the iterative solution of the integral equation.
\[ \int_0^1 (x+y)^\frac{1}{2} \left[ \varphi(y) \right]^\frac{3}{2} \, dy = \varphi(x) \] and find bounds on \( \varphi(x) \).

8. (a) Find first three iterates of the solution of
\[ \varphi(x) = \lambda \int_0^x \sin(xy) \varphi(y) \, dy + 1 \]

(b) Show that the solution of
\[ \varphi(x) - \int_0^x e^{\alpha y} \varphi(y) \, dy = 1 - x^{-1} (e^x - 1) \] is \( \varphi(x) = 1 \).

Also, find approximation to \( \varphi \left( \frac{1}{4} \right) \) and \( \varphi \left( \frac{3}{4} \right) \) when \( \varphi(x) \) is determined by the integral equation
\[ \varphi(x) - \int_0^x e^{\alpha y} \varphi(y) \, dy = 1 - x^{-1} (e^x - 1). \]
1. (a) Explain the transportation problem with an example.
(b) State and prove optimality criterion for the standard maximization problem.
(c) Using equilibrium theorem show that 
   \[ \{\varepsilon_1 = 4, \varepsilon_2 = 1\} \] is an optimal solution of the problem

Maximize: \[ \varepsilon_1 - \varepsilon_2 \]
Subject to 
\[ -2\varepsilon_1 + \varepsilon_2 \leq 2 \]
\[ \varepsilon_1 + 2\varepsilon_2 \leq 2 \]
\[ \varepsilon_1 + \varepsilon_2 \leq 5 \]

2. (a) Define row rank and column rank of an \( m \times n \) matrix \( A \) and show that for any matrix \( A \), row rank and column rank of \( A \) are equal.
(b) Show that the equation \[ \begin{cases} \varepsilon_1 + 3\varepsilon_2 - 5\varepsilon_3 = 2 \\ \varepsilon_1 + 4\varepsilon_2 - 7\varepsilon_3 - 3 \end{cases} \] have no non negative solution.
(c) Show that if \( A \) is an \( m \times n \) matrix then the set of all solutions of \( x A \leq 0 \) is a finite cone.

3. (a) Transform the following standard problem into canonical

Maximize \[ 3x_1 - x_2 + 2x_3 \]
\[ -x_1 + 2x_2 - x_2 \leq 3 \]
\[ 2x_1 - x_2 + x_3 \leq -4 \]
\[ x_1 + 3x_2 - 2x_3 \leq 5 \]
\[ n_i \geq 0 \] for \( i = 1, 2, 3 \)

(b) State and prove fundamental duality theorem.
(c) Using canonical equilibrium theorem prove that \( \{\varepsilon_1 = 1, \varepsilon_2 = 1, \varepsilon_i = y_2, \varepsilon_4 = 0\} \) is an optimal solution to the problem.
maximize : $2\varepsilon_1 + 4\varepsilon_2 + \varepsilon_3 + \varepsilon_4$

Subject to $\varepsilon_1 + 3\varepsilon_2 + \varepsilon_4 = 4$

$2\varepsilon_1 + \varepsilon_2 = 3$

$\varepsilon_2 + 4\varepsilon_3 + \varepsilon_4 = 3$

$\varepsilon_i \geq 0 \forall i = 1, 2, 3, 4$

4. (a) Solve the following equation using replacement operation.

$2\varepsilon_1 + 3\varepsilon_2 - \varepsilon_3 = 1$

$\varepsilon_1 + 2\varepsilon_3 = -2$

$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 2$

(b) Solve the following problem by simplex method

Maximize $8\varepsilon_1 + 19\varepsilon_2 + 7\varepsilon_3$

Subject to $3\varepsilon_1 + 4\varepsilon_2 + \varepsilon_3 \leq 25$

$\varepsilon_1 + 3\varepsilon_2 + 3\varepsilon_3 \leq 50$

$\varepsilon_i \geq 0 \forall i = 1, 2, 3$

5. (a) Define a capacitated network, flow in a network, source and sink for a flow and a cut network. State maxi flow and mini cut theorem.

(b) Prove Max flow - min cut theorem.

(c) Solve the following assignment problem.

<table>
<thead>
<tr>
<th></th>
<th>$J_1$</th>
<th>$J_2$</th>
<th>$J_3$</th>
<th>$J_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_1$</td>
<td></td>
<td></td>
<td>$X$</td>
<td>$X$</td>
</tr>
<tr>
<td>$I_2$</td>
<td>$X$</td>
<td></td>
<td></td>
<td>$X$</td>
</tr>
<tr>
<td>$I_3$</td>
<td>$X$</td>
<td>$X$</td>
<td></td>
<td>$X$</td>
</tr>
<tr>
<td>$I_4$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td></td>
</tr>
</tbody>
</table>
6. (a) Solve the assignment problem whose rating matrix is as follows.

\[
\begin{array}{cccccc}
& J_1 & J_2 & J_3 & J_4 & J_5 \\
I_1 & 12 & 9 & 10 & 3 & 8 \\
I_2 & 6 & 6 & 2 & 2 & 9 \\
I_3 & 6 & 8 & 10 & 11 & 9 \\
I_4 & 6 & 3 & 4 & 1 & 1 \\
I_5 & 11 & 1 & 10 & 9 & 12 \\
\end{array}
\]

(b) State and prove feasibility theorem for a transhipment problem.

7. (a) Solve the transportation problem.

\[
\begin{array}{cccc|c}
& M_1 & M_2 & M_3 & M_4 & \text{Supply} \\
P_1 & 4 & 4 & 9 & 3 & 3 \\
P_2 & 3 & 5 & 8 & 8 & 5 \\
P_3 & 2 & 6 & 5 & 7 & 7 \\
\text{Demand} & 2 & 5 & 4 & 4 & \\
\end{array}
\]

(b) Define two person zero sum game. Explain the games of odds and even's and 2. Matching pennies and write down payoff matrix for each game.

8. (a) Define saddle point and minimax and state and prove minimax theorem.

(b) Construct a game with value ‘0’ having the sets \(\overline{X} = (x_1, x_2, x_3)\), \(\overline{Y} = (y_1, y_2)\)
where \(\overline{x}_1 = (1, 0, 0)\), \(\overline{x}_2 = (y_2, y_2, 0)\), \(\overline{x}_3 = (y_3, y_3, y_3)\) and \(\overline{y}_1 = (1, 0)\), \(\overline{y}_2 = (y_2, y_2)\).
1. (a) Write a FORTAN programme to generate prime numbers between 1 and 100.
   (b) Write a function subprogram to sort numbers in ascending (use array in the program)

2. (a) Explain FORMAT description for READ and PRINT statements.
     (b) Explain the rules to be followed in implementing DO loops.

3. (a) Using $\sin(0.1) = 0.09983$ and $\sin(0.2) = 0.19867$ find an approximate value of $\sin(0.15)$ by Lagrange interpolation method. Obtain a bound on the truncation error.
     (b) Derive Simpson's rule

4. (a) Obtain the unique polynomial of degree 2 or less, such that
     \[ f(0) = 1, f(1) = 3, f(3) = 55 \] by using Newton's divided difference interpolation.
     (b) Calculate $\int_{0.5}^{0.5} \frac{x}{\sin(x)} dx$ using Romberg integration with step size $h = 1/16$.

5. (a) Use 4th order Runge Kutta method to solve the initial value problem
     \[ u' = -2tu^2, u(0) = 1 \] on $[0, 1]$ with step length as 0.1
     (b) Use Numerov method to solve the BVP $u'' = u + x, u'(0) = u(1) = 0$ with $h = 1/1$.

6. (a) Solve \[ y' - y \sin x = \cos x \] subject to $y(0) = 0$ using Taylor's series method.
     (b) Find the value of solution at $x = 0.3$ for the IVP $y' = 3e^y + 2y, y(0) = 0$ by using Adams Bashforth method of order 2 with $h = 0.1$.

7. (a) Write a FORTAN program to compute solution of an IVP using Euler's method
     (b) Write a FORTAN program to compute solution of an IVP using Predictor-Corrector method.

8. (a) Write a FORTAN program to compute solution of an IVP using Runge Kutta method.
     (b) Write a FORTAN program to evaluate an integral using Romberg Integration.
ANDHRA UNIVERSITY
SCHOOL OF DISTANCE EDUCATION
M.A. / M.SC., Mathematics (Final Year)
ASSIGNMENTS
Universal Algebra
Answer any five of the following. All question carry equal marks. 

5x3 = 15 Marks

1. (a) Show that a lattice $L$ is modular if and only if for any

$$x, y, z \in L; (x \land y) \lor (y \land z) = y \land ((x \land y) \lor z)$$

(b) Prove that a lattice $(L, \lor, \land)$ is a non distributive lattice if and only if either $M$, or $N$, can be embedded in $L$.

2. (a) Let $A$ be a non empty set. Then prove that the posets

$$(\pi(A), \leq) \text{ and } (Eq(A), \subseteq) \text{ are order isomorphic.}$$

(b) Prove that every complete lattice $L$ is isomorphic to the lattice $L_c$ of closed subsets of some set $A$ with a closure operator $C$.

3. (a) If $L$ is an algebraic lattice, then prove that $L \cong \text{Sub}(A)$ for some algebra $A$.

(b) Let $C$ be an $n$-ary closure operator on a set $S$ and $n \geq 2$. Suppose $i, j \in I, B(C)$ such that $i < j$ and $K \not\in I, B(C)$ for all $i < k < j$. Then prove that $j - 1 \leq n - 1$ and if $n = 2$, then prove that $I, B(C)$ is a sequence of consecutive natural numbers.

4. (a) Let $\theta$ and $\phi$ be equivalence relations on a set $A$. Then prove that the following are equivalent to each other

1) $\theta \circ \phi \subseteq \phi \circ \theta$

2) $\theta \circ \phi = \phi \circ \theta$

3) $\theta \circ \phi = \theta \lor \phi$

(b) State and prove the correspondence theorem.

5. (a) Prove that an algebra $A$ is subdirectly irreducible if and only if either it is trivial or there is a least non zero (non-diagonal) congruence on $A$. 
(b) Prove that every algebra $A$ is a subdirect product of subdirectly irreducible algebras.

6. (a) Prove that $V = HSP$.
   (b) Prove that any variety with a nontrivial member contains a non trivial simple algebra.

7. (a) Prove that $K$ is an equational class if and only if $K$ is a variety.
   (b) Prove that a variety $V$ of type $J$ is congruence permutable if and only if there is a term $P(x, y, z)$ such that $V |\models P(x, x, y) \approx y$ and $V |\models P(x, y, y) \approx x$.

8. (a) Let $B$ be a boolean algebra and $\theta$ be a binary relation on $B$. Then prove that $\theta$ is a congruence on $B$ if and only if $0/\theta$ is an ideal and, for any $a, b \in B(a, b) \in \theta$ if and only if $a + b \in 0/\theta$.
   (b) Let $P$ be a primal algebra. Then prove that
   
   $$V(P) = \mathcal{I}\left\{ P(B) / B \text{ is a Boolean algebra} \right\}. $$